Closing *today:* HW6A,6B,6C (7.3,7.4,7.5)
Closing next *Wed*: HW7A,7B (7.5,7.7,7.8)
How would you ***start*** these?

Note: Mon. is a university holiday (no class)
Exam 2 is next **Thursday** (Feb. 22nd)
Covers 6.4, 6.5, 7.1-7.5, 7.7, 7.8

1. $\displaystyle\int \frac{x^2+7}{x^2(3-x)}\,dx$

2. $\displaystyle\int \sqrt{x}\ln(x)dx$

3. $\displaystyle\int \frac{1}{(x^2+6x+13)^{3/2}}\,dx$

4. $\displaystyle\int \tan^{-1}(x)\,dx$

5. $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$

6. $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

7. $\displaystyle\int \frac{\sqrt{x}}{x-9}\,dx$

8. $\displaystyle\int \tan^4(x)\sec^4(x)\,dx$

9. $\displaystyle\int x\sqrt{4-x}\,dx$

7.7 Approximating Integrals

We have learned how to integral some important situations. ***But*** many, many, many integrals CANNOT be done with any of our methods. So, in a great many applications, we have to approximate!

To approximate $\int_a^b f(x)\,dx$

1. Compute $\Delta x = \frac{b-a}{n}$.
   Label the tick marks: $x_i = a + i\Delta x$
2. Use an approximation method:

$$L_n = \Delta x[f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad \text{(Left endpoint)}$$
$$R_n = \Delta x[f(x_1) + f(x_2) + \cdots + f(x_n)] \quad \text{(Right endpoint)}$$
$$M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad \text{(Midpoint)}$$

New - Trapezoid Rule: (all the "middle terms" are multiplied by 2)

$$T_n = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

New - Simpson's Rule: *n* must be even! (Alternating multiplying middle terms by 4 and 2)

$$S_n = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example: Using n = 4 subdivisions, estimate

$$\int_0^4 \sqrt{100 - x^3}\,dx$$

- ***Step 1***: $\Delta x = \frac{4-0}{4} = 1.$ $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$
- ***Step 2****:* Here is each method:

$$(1)\left[\sqrt{100-(0)^3}+\sqrt{100-(1)^3}+\sqrt{100-(2)^3}+\sqrt{100-(3)^3}\right] \approx 38.0855 = L_4$$

$$(1)\left[\sqrt{100-(1)^3}+\sqrt{100-(2)^3}+\sqrt{100-(3)^3}+\sqrt{100-(4)^3}\right] \approx 34.0855 = R_4$$

$$(1)\left[\sqrt{100-(0.5)^3}+\sqrt{100-(1.5)^3}+\sqrt{100-(2.5)^3}+\sqrt{100-(3.5)^3}\right] \approx 36.5672$$

NEW – Trapezoid rule

$$\frac{1}{2}(1)\left[\sqrt{100-(0)^3}+2\sqrt{100-(1)^3}+2\sqrt{100-(2)^3}+2\sqrt{100-(3)^3}+\sqrt{100-(4)^3}\right]$$

$$T_4 \approx 36.0855$$

NEW – Simpson's rule

$$\frac{1}{3}\cdot(1)\left[\sqrt{100-(0)^3}+4\sqrt{100-(1)^3}+2\sqrt{100-(2)^3}+4\sqrt{100-(3)^3}+\sqrt{100-(3)^3}\right]$$

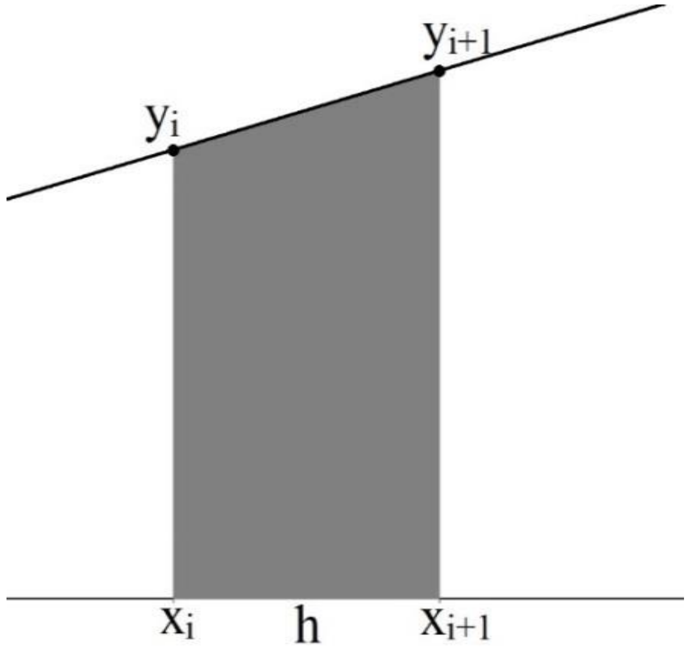
$$S_4 \approx 36.3863$$

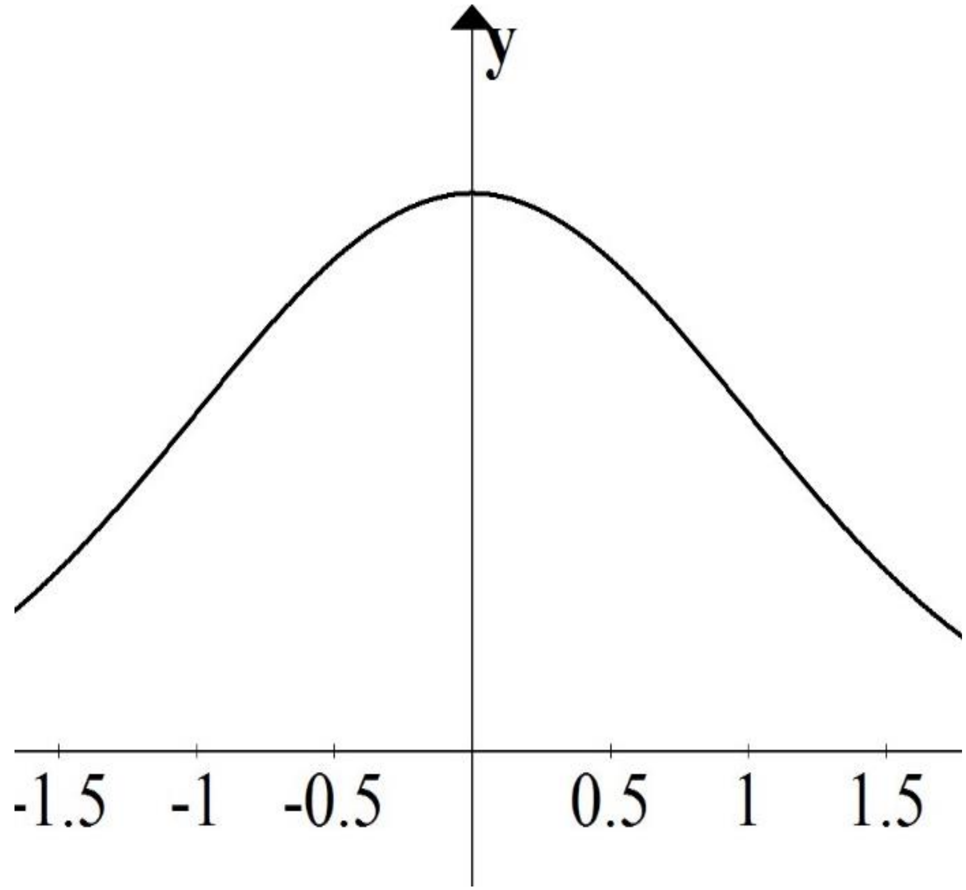
"Actual" Value (to 8 places after the decimal) ≈ 36.40897795

7.7 Derivation Notes

Trapezoid Rule:

$$\text{Shaded Area} = \frac{h}{2}(y_i + y_{i+1})$$

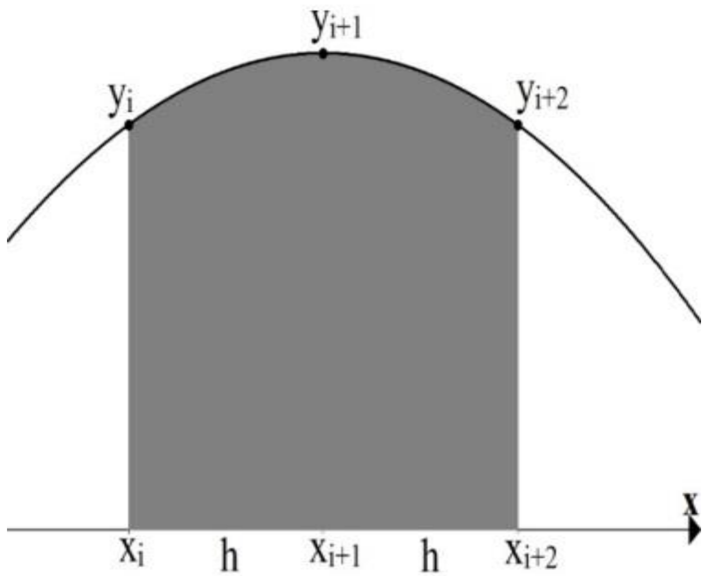


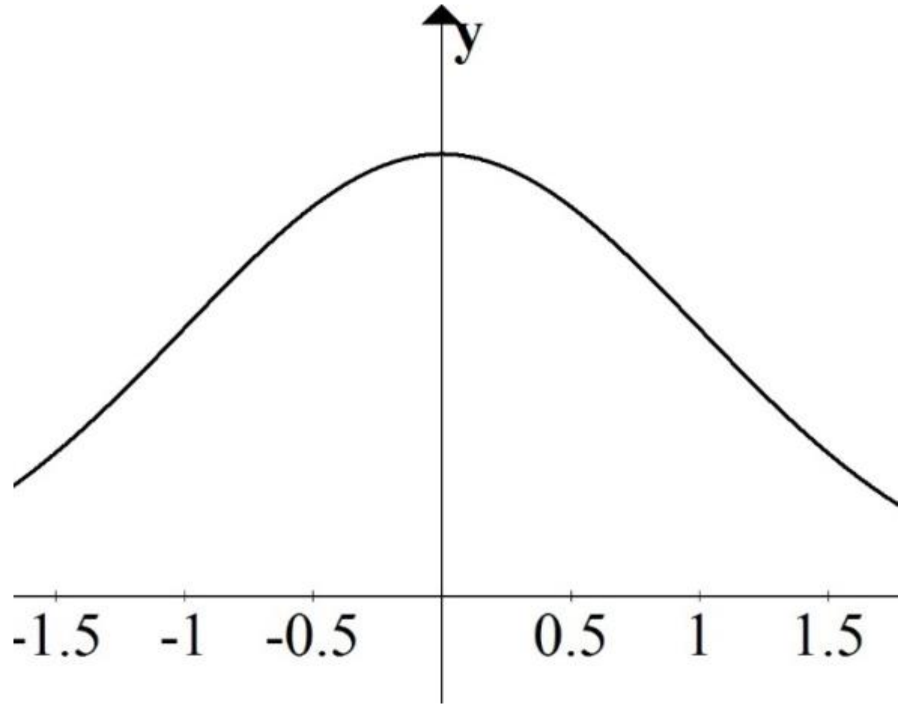


Simpson's Rule:

The ***parabola***, $y = ax^2 + bx + c$, thru the three indicated points, satisfies

$$\text{Shaded Area} = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$





Example: With n = 4, use both new methods to approximate (just set up):

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx$$

$\Delta \boldsymbol{x} =$ $\quad, \boldsymbol{x_0} =$ $\quad, \boldsymbol{x_1} =$ $\quad, \boldsymbol{x_2} =$ $\quad, \boldsymbol{x_3} =$ $\quad, \boldsymbol{x_4} =$

$$T_4 = \frac{1}{2}\Delta x[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$S_4 = \frac{1}{3}\Delta x[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

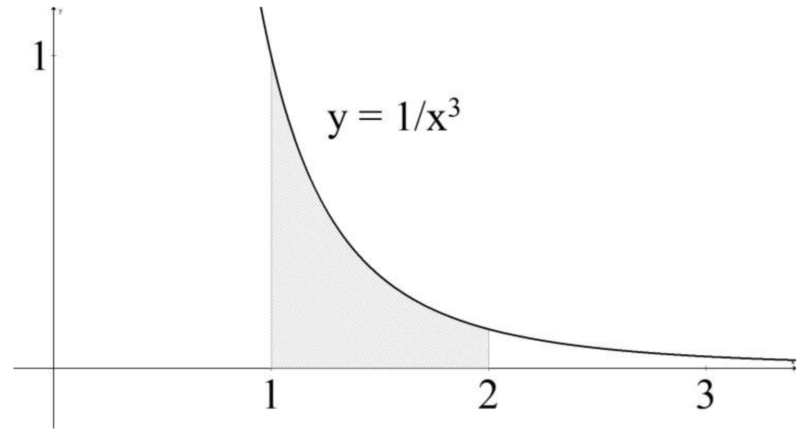
7.8 Improper Integrals

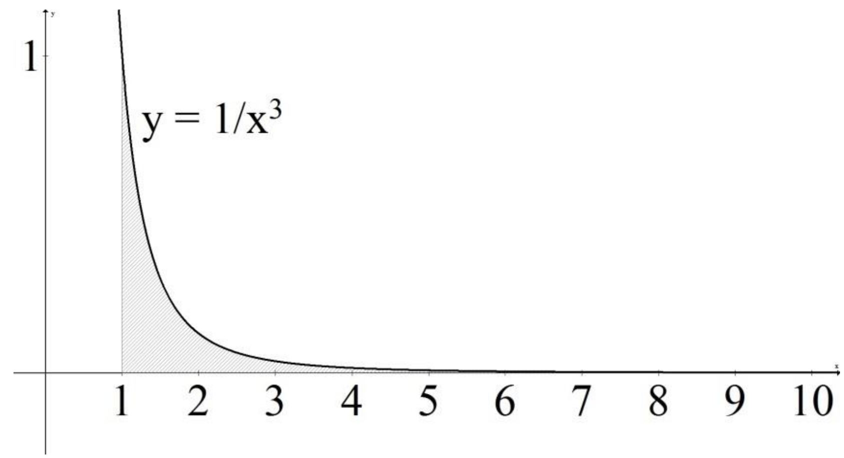
Motivation:

Consider the function $f(x) = \frac{1}{x^3}$.

Compute the area under the function…

1. …from $x = 1$ to $x = t$
2. …from $x = 1$ to $x = 10$
3. …from $x = 1$ to $x = 100$





Def'n: *Improper type 1 - infinite integral of integration*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.
Otherwise, we say it *diverges*.

Example:

$$\int_1^\infty \frac{1}{x^3}dx =$$

Example:

$$\int_{-1}^{\infty} e^{-2x}dx =$$

Example:

$$\int_{1}^{\infty} \frac{1}{x}dx =$$

Def'n:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r \to -\infty} \int_{r}^{0} f(x)dx + \lim_{t \to \infty} \int_{0}^{t} f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

Def'n: *Improper type 2 - infinite discontinuity*

If $f(x)$ has a discontinuity at x = a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

If $f(x)$ has a discontinuity at x = b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.

Otherwise, we say it *diverges*.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}} dx =$$

Example:

$$\int_0^2 \frac{x}{x-2}\,dx =$$

If $f(x)$ has a discontinuity at $x = c$ which is **between** a and b, then

$$\int_a^b f(x)dx = \lim_{r\to c^-}\int_a^r f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

$$\int_0^\pi \frac{1}{\cos^2(x)}dx =$$

Limits Refresher

1. If stuck, plug in values "near" t.
2. Know your basic functions/values:

$$\lim_{t\to\infty} \frac{1}{t^a} = 0, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \frac{1}{e^{at}} = 0, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} t^a = \infty, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \ln(t) = \infty.$$

$$\lim_{t\to 0^+} \ln(t) = -\infty.$$

3. For indeterminant forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t\to 1} \frac{t^2 + 2t - 3}{t - 1} =$$

$$\lim_{t\to\infty} \frac{\ln(t)}{t} =$$

$$\lim_{t\to\infty} t^2 e^{-3t} =$$

Aside:

A few general notes on **comparison**:

Suppose you have two functions f(x) and g(x) such that $0 \le g(x) \le f(x)$ for all values.

(a) If $\int_1^\infty f(x)dx$ converges,
then $\int_1^\infty g(x)dx$ converges.

(b) If $\int_1^\infty g(x)dx$ diverges,
then $\int_1^\infty f(x)dx$ diverges.

You can verify that

$$\int_1^\infty \frac{1}{x^p}\,dx\,, \qquad \text{converges for } p > 1.$$

$$\int_1^\infty e^{px}\,dx\,, \qquad \text{converges for } p < 0.$$

And you can compare off of these to sometimes quickly tell is something is converging or diverging (without calculating anything)